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Analysis of Local Stability for Discrete Lotka-Volterra Models

Análise da Estabilidade Local para Modelos Discretos Lotka-Volterra

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ABSTRACT

This paper applies the stability theory of discrete systems to a predator–prey model with a specific structure, formulated directly in discrete time. This approach offers didactic and computational advantages for modeling ecological systems with non-overlapping generations, contrasting with methods that discretize continuous models. This direct formulation captures the inherently discrete nature of ecological monitoring and non-overlapping generations, while presenting particular analytical challenges. Through linearization and spectral analysis, we obtain explicit stability conditions for the system's three equilibria: total extinction, which is always unstable; predator exclusion; and coexistence, whose local behaviors depend on conditions among biotic parameters. The results provide practical criteria for predicting population persistence, offering a foundation for applied studies in control and conservation.

keywords Lotka-Volterra, mathematical modeling, population dynamics, stability of discrete systems

RESUMO

Este trabalho aplica a teoria de estabilidade de sistemas discretos a um modelo predador-presa com estrutura específica, formulado diretamente em tempo discreto. Esta abordagem oferece vantagens didáticas e computacionais para a modelagem de sistemas ecológicos com gerações não sobrepostas, contrastando com métodos que discretizam modelos contínuos. Esta formulação direta captura a natureza inerentemente discreta do monitoramento ecológico e de gerações não sobrepostas, ao mesmo tempo que apresenta desafios analíticos particulares. Por meio de linearização e análise espectral, obtemos condições explícitas de estabilidade para os três equilíbrios do sistema: extinção total, que é sempre instável; exclusão de predadores e coexistência, cujos comportamentos locais dependem de condições entre os parâmetros bióticos. Os resultados fornecem critérios práticos para prever a persistência populacional, oferecendo uma base para estudos aplicados em controle e conservação.

palavras-chave Lotka-Volterra, modelagem matemática, dinâmica populacional, estabilidade de sistemas discretos

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Introduction

In predator-prey model studies, a pertinent question is: how can we translate the mathematical properties of these equations, often elegant and continuous, into practical answers for ecologists dealing with limited data? This work applies discrete systems stability theory to analyze a predator-prey model formulated directly in discrete time. This approach offers didactic and computational advantages while providing a more natural framework for systems with non-overlapping generations and ecological monitoring at fixed intervals, contrasting with common approaches that discretize continuous models. This sensitivity to discretization is well documented in recent studies, where numerical step size and parameter values may induce qualitative changes in the system dynamics (Kekulthotuwage Don et al., 2023).

This study stems from the interest in deepening theoretical aspects and applications of discrete models in population ecology; this approach involves the qualitative study of difference equations (Diniz, 2011). Although ecological processes occur in continuous time, practical research in population dynamics often deals with discrete data: seasonal censuses, annual monitoring, or non-overlapping generations. This discrete modeling has historical roots: the logistic equation of Verhulst (1838), proposed as a counterpoint to the unlimited growth of Malthus (1826), was later popularized by May (1976) in its discrete version, which revealed its potential to model complex behaviors such as bifurcations and chaos in simple systems.

Contemporary studies have significantly expanded this field: Braverman and Kinzebulatov (2006) established rigorous stability criteria for perturbed Ricker discrete models, while Seno (2008) demonstrates how management strategies can lead to counterintuitive effects in discrete population dynamics. These advances have not only consolidated the theory of one-dimensional systems, but have also paved the way for the analysis of more complex interactions. More recent approaches have also investigated stability in Lotka–Volterra systems from global and structural perspectives, including the use of invasion graphs and geometric methods (Almaraz et al., 2024).

Additionally, discrete predator–prey models incorporating non-linear functional responses and semi-discretization techniques have been studied in connection with stability and bifurcation phenomena (Lv & Li, 2024). In the national context, Luiz et al. (2022) investigate a predator–prey model formulated as a system of partial differential equations of Telegraph type and analyze the stability and convergence of finite-difference discretization schemes. Although our model is formulated directly in discrete time, this work is included to illustrate how discretization of continuous predator–prey systems may affect stability properties and numerical behavior.

In this context, discrete versions of the Lotka-Volterra model emerge as powerful tools for capturing predator-prey dynamics in systems with discrete temporal observations (Din, 2013; Khaliq et al., 2022). While various discrete formulations exist in the literature, it is crucial to distinguish the approach adopted here. Previous works, such as Din (2013), often investigate models incorporating logistic growth terms for both predator and prey populations.

In contrast, this work adopts a distinct conceptual framework. For the prey, we use the classical form of the discrete logistic equation, as presented by Smith in Fisher et al. (1979), suitable for populations with non-overlapping generations. For the predators, we assume a simpler growth model, without a logistic term, where the dynamics are governed essentially by the intrinsic mortality rate and predation success. This choice reflects ecological scenarios where predators, being typically less abundant and more mobile, do not experience strong enough intraspecific competition to impose an explicit carrying capacity in the short term.

Thus, the discrete Lotka-Volterra model we study here serves as a tool to analyze observable scenarios in the interaction of two species: x_n (prey) with logistic growth and y_n (predator):

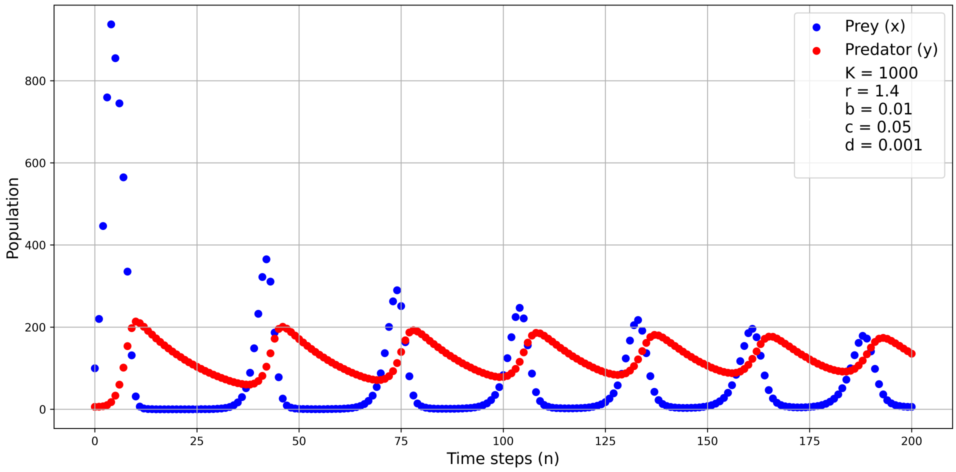
$$\begin{cases} x_{n+1} = x_n \left[1 + r\left(1 - \frac{x_n}{K}\right)\right] - b x_n y_n, \\ y_{n+1} = (1-c) y_n + d x_n y_n, \end{cases} \tag{1}$$

where r is the prey growth rate, K is the carrying capacity, b and d are species interaction parameters, and c is the predator decay rate.

All biotic parameters are strictly positive, and in the absence of predators ($y_n = 0$), the system of equations (1) reduces to the discrete logistic model presented by Maynard Smith in 1968, applied to populations with non-overlapping generations Fisher et al. (1979).

Figure 1 shows the temporal evolution of these populations, obtained by numerical simulation of the discrete-time system (1). The simulation was performed by directly iterating the system's equations with initial conditions $x_0 = 100$ and $y_0 = 6$. The parameter values $r = 1.4$, $b = 0.01$, $c = 0.05$, and $d = 0.001$ were selected to satisfy the coexistence equilibrium conditions derived in this work, and the simulation was run over 200 iterations.

Figure 1 - Oscillations of the populations x_n and y_n in the predator-prey system (1).



The notion of equilibrium state is fundamental to the study of discrete systems dynamics. Across various fields such as biology, economics, physics, and engineering, it is often desirable that all states or solutions of a discrete system converge to an equilibrium state. This behavior lies at the heart of stability theory, a field of great importance to scientists and engineers. In this context, a system is considered stable if, when subjected to a small change in its initial conditions, it does not deviate indefinitely but rather maintains or returns to predictable behavior.

The objective is to establish conditions among the biotic parameters that imply stability of the equilibria for system of equation (1), which are identified as:

- Total extinction (E_1),
- Predator extinction (E_2),
- Coexistence (E_3).

A systematic theoretical foundation is then established to support future applied investigations. The main contributions are as follows:

- **A self-contained stability analysis:** We apply classical theory to obtain explicit stability conditions for the three equilibria in the discrete-time Lotka-Volterra model given by the system of equations (1).
- **Analysis of coexistence:** We provide stability results for the coexistence equilibrium (E_3) using the trace-determinant method, revealing its parametric sensitivity.
- **A bridge between theory and application:** The primary value of this work lies in organizing these mathematical results, establishing a foundation for future research in control, conservation, and numerical simulation for system of equation (1).

The results, obtained via linearization and spectral analysis of the Jacobian matrix, provide a mathematical methodology for interpreting dynamics observed in real systems with discrete sampling. This is the first step for future applications in control and management. In summary, to analyze the stability of the discrete Lotka-Volterra model, equation (1), we adopt a three-step approach: identification of the equilibria, linearization of the system around these points, and application of stability criteria based on the eigenvalues of the Jacobian matrix. This strategy is typical for dynamical systems Saber (2007) and Krabs and Pickl (2010). The results reveal a subtle picture: while the equilibrium E_1 is unconditionally unstable, the equilibria E_2 and E_3 require fine adjustments between growth and interaction to guarantee stability.

Materials and methods

In the study of stability for nonlinear discrete systems, a common approach is the linearization of the system around equilibrium points. This method involves approximating the nonlinear equations by linear equations, which simplifies the analysis of the system's local behavior. Thus, stability analysis through linearization becomes an essential tool for understanding the behavior of nonlinear systems and evaluating their ability to return to an equilibrium state.

The theoretical foundation that follows, necessary for the local stability analysis, is primarily based on the works of Krabs and Pickl (2010) and Saber (2005, 2007). The definitions and theorems have been adapted in notation and scope to better suit the objectives of this research, particularly for the analysis of system (1). The following definitions and theorems consist of classical results from discrete dynamical systems theory, reformulated in a notation that better suits the context and objectives of the present work.

Local stability and linearization

To analyze the local stability of the system, we first recall key mathematical concepts and formal definitions that will be used throughout work, in which a basic and standard concept in difference equations theory is introduced, for which equivalent formulations can be found in the introductory chapters of Saber (2005, 2007), as well as a classical concept adapted from Saber (2005). Moreover, the linearized system is formalized following the standard procedure exemplified in Saber (2005, 2007), and the classical concept of a saddle point in continuous systems is adapted to discrete systems, following the intuition established in Saber (2007) for hyperbolic points.

Furthermore, we state a theorem essential for the analysis of stability; for the complete proofs of the constituent results, see Theorem 1.5 and its corollary in Krabs and Pickl (2010), pp. 47, 49.

Definition 1 (Equilibrium Point)**.** An element E in the domain of a function $f \in C^1$ (continuously differentiable on an open set) is called an equilibrium point or state for the system $x_{n+1} = f(x_n)$ if

$$E = f(E). \tag{2}$$

Equation (2) states that E is a fixed point of the function f. Thus, an element $E = (x^*, y^*) \in \mathbb{R}^2$ is an equilibrium point for system of equations (1) if,

$$\begin{cases} x^*\left(r\left(1-\frac{x^*}{K}\right)-by^*\right) &= 0, \\ y^*\left(-c+dx^*\right) &= 0. \end{cases} \tag{3}$$

By solving system of equations (3), we identify the equilibria:

- Total extinction: $E_1 = (0, 0)$.
- Predator extinction: $E_2 = (K,\, 0)$,
- Coexistence: $E_3 = \left(\frac{c}{d},\, \frac{r}{bK}\left(K-\frac{c}{d}\right)\right)$ with $K > \frac{c}{d}$.

Being compatible with biological hypotheses, we consider positive equilibria.

Local Stability means that, if we take an initial value $(x_0,\, y_0)$ sufficiently close to $E = (x^*, y^*)$, the subsequent iterates $(x_n,\, y_n)$ will remain close to $(x^*,\, y^*)$.

Definition 2. Let E be an equilibrium point of the system $x_{n+1} = f(x_n)$.

1. E is called locally stable if, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $||x_0 - E|| < \delta$ then $||x_n - E|| < \varepsilon$ for all , $\forall n \in \mathbb{N}$.
2. E is called unstable if it is not stable.

3. A locally stable equilibrium E is called asymptotically stable if there exists $\eta > 0$ such that if $||x_0 - E|| < \eta$ then $\lim\limits_{n\to\infty} ||x_n - E|| = 0$.

To analyze the local stability of nonlinear systems of equations (1), it is appropriate to consider linearization around equilibrium points, which simplifies the analysis by approximating nonlinear equations with linear equations. This technique is essential for understanding the system's local behavior and its ability to return to a predictable state.

Using vector notation, the system of equations (1) can be written as

$$(x_{n+1},\, y_{n+1}) = (f_1(x_n,\, y_n), f_2(x_n,\, y_n))\,, \tag{4}$$

as shown in equation (4), where

$$f_1(x,\, y) = x\left[1 + r\left(1 - \frac{x}{K}\right)\right] - bxy \quad \text{and} \quad f_2(x,\, y) = (1-c)y + dxy. \tag{5}$$

These component functions, defined in (5), capture the prey and predator dynamics, respectively. Thus, the Jacobian matrix of the vector field $f(x,\, y)$ is defined as

$$\mathbf{J}_f(x,\, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{bmatrix}. \tag{6}$$

The Jacobian matrix (6) is central to our stability analysis. Next, we define the linearized system.

Definition 3 (Linearized System)**.** Let (x^*, y^*) be an equilibrium of system (1). The Linearized System around this equilibrium is defined by the following linear system:

$$\begin{cases} x_{n+1} &= a_{11}x_n + a_{12}y_n \\ y_{n+1} &= a_{21}x_n + a_{22}y_n \end{cases} \tag{7}$$

where $[a_{ij}]_{2\times 2} = \mathbf{J}_f(x^*, y^*)$.

If $(x^*,\, y^*)$ is an equilibrium point under the conditions of Definition 3, and denoting by $\mathbf{X}_n = (x_n,\, y_n)$, $\mathbf{J} = [a_{ij}]_{2\times 2}$, we can rewrite the Linearized System as

$$\mathbf{X}_{n+1} \;=\; \mathbf{J}\mathbf{X}_n, \tag{8}$$

where

$$\mathbf{J} = \begin{bmatrix} 1 + r(1 - \frac{2x^*}{K}) - by^* & -bx^* \\ dy^* & 1 - c + dx^* \end{bmatrix}. \tag{9}$$

The local stability of an equilibrium can be determined by analyzing the linearized system. The Theorem 1 provides a criterion based on the eigenvalues of the Jacobian matrix (9).

Theorem 1 (Spectral Criterion for Local Stability)**.** *Let X be an open subset of $\mathbb{R}^k$ and $x^* \in X$ a fixed point of a function $f : X \to X$, with $f \in C^1$, and let $J_f(x^*)$ be the Jacobian matrix of f evaluated at x^*. Denote by $\varrho(J_f(x^*))$ the spectral radius of $J_f(x^*)$, i.e., the largest modulus of the eigenvalues of $J_f(x^*)$. Then, the following statements hold:*

a) ***(Asymptotic Stability)*** *If $\varrho(J_f(x^*)) < 1$, and $x_n = f^n(x_0)$, then $\lim\limits_{n\to\infty} ||x_n - x^*|| = 0$. This establishes an asymptotically stable equilibrium.*

b) ***(Source-Type Instability)*** *If $J_f(x^*)$ is invertible, $x_n = f^n(x_0)$, and every eigenvalue of $J_f(x^*)$ has modulus greater than 1, then there exists $\delta > 0$ such that if $||f^k(x_0) - x^*|| < \delta$ for $0 \leq k \leq n-1$, then $\lim\limits_{n\to\infty} ||x_n - x^*|| = +\infty$. This establishes an unstable equilibrium point of source type.*

Theorem 1 is a restatement of the classical result from Krabs and Pickl (2010). This approach is more natural for discrete systems in $\mathbb{R}^k$, and is directly applicable to ecological contexts where the behavior of iterates $x_n = f^n(x_0)$ provides clear dynamic interpretation regarding equilibria.

Definition 4 (Saddle Point)**.** If $J(E)$ has eigenvalues λ_1, λ_2 with $|\lambda_1| < 1 < |\lambda_2|$, then E is an unstable equilibrium of Saddle Point type.

This definition is consistent with the previous criteria based on spectral radius. Indeed, it suffices for a single eigenvalue with modulus greater than 1 to make the equilibrium point unstable, even if the other eigenvalue lies within the unit disk. The saddle point case precisely illustrates this situation: the presence of a dominant eigenvalue $|\lambda_2| > 1$ guarantees local instability of the equilibrium. A broader theoretical justification can be provided by the Hartman-Grobman Theorem, which in the context of discrete systems ensures the validity of linear analysis around hyperbolic points of nonlinear systems Saber (2007).

Results and discussion

With the theoretical foundation established, we now turn to the study of the equilibrium points' behavior in the discrete Lotka-Volterra system. An equilibrium point $E = (x^*,\, y^*)$ of system (1) is, by definition, a solution of system (3). Next, we will analyze each of these points to better understand their local characteristics.

We emphasize that our analysis will be conducted from the perspective of hyperbolic equilibrium points, that is, those for which the eigenvalues of the associated Jacobian satisfy $|\lambda_i| \neq 1$. This restriction allows us to employ classical local stability criteria, since non-hyperbolic cases require more in-depth study.

Trivial equilibrium

The equilibrium $E_1 = (0, 0)$ corresponds to the scenario of complete extinction, where both populations are absent. The Jacobian matrix, equation (9), at this point takes a simple diagonal form:

$$\mathbf{J}_1^* = \begin{bmatrix} 1+r & 0 \\ 0 & 1-c \end{bmatrix} \tag{10}$$

whose eigenvalues are $\lambda_1 = 1 + r$ and $\lambda_2 = 1 - c$.

Since $r > 0$ by biological hypothesis, it follows that $|\lambda_1| = 1 + r > 1$. This condition, as previously explained, already allows us to conclude that the equilibrium E_1 is unstable, regardless of the predator mortality rate c. The nature of this instability depends on the behavior of λ_2, that is, on the parameter c. Thus, the instability can be classified according to two criteria:

1. When $c > 2$: Both eigenvalues of the Jacobian matrix, equation (10), have modulus greater than one, characterizing E_1 as an unstable **source** according to item b) of Theorem 1.
2. When $0 < c < 2$: We have $|1 - c| < 1$, thus $|\lambda_2| < 1 < |\lambda_1|$ reveals a **saddle point**, following Definition 4.

Predator extinction equilibrium

This equilibrium corresponds to prey survival at their carrying capacity K, with predator extinction. The Jacobian matrix, equation (9), at $E_2 = (K,\, 0)$ is upper triangular:

$$J_2^* = \begin{bmatrix} 1-r & -bK \\ 0 & 1-(c-dK) \end{bmatrix} \tag{11}$$

whose eigenvalues are $\lambda_1 = 1 - r$ and $\lambda_2 = 1 - (c - dK)$.

Note that in this case, the eigenvalues of the Jacobian matrix, equation (11) depend on a more complex interaction between the model parameters. Beyond r and c, they explicitly involve b, d and the carrying capacity K, making the analysis more sensitive to system variations. This richer parametric structure requires careful examination of possible combinations influencing the local stability of equilibrium E_2.

Depending on these parameter combinations, three types of local behavior can occur: Asymptotic stability, Source-type instability, and Saddle point.

Asymptotic stability

Applying item (a) of Theorem 1 we obtain that E_2 is an asymptotically stable equilibrium if,

$$0 < r < 2 \quad \text{and} \quad dK < c < 2 + dK.$$

From a biological perspective, this means the populations remain stable at the point $E_2 = (K, 0)$ when the prey growth rate r is moderate and predators are ineffective at invading the system. That is, the predator mortality rate exceeds the benefits gained from predation at maximum prey density. Note also that the carrying capacity lies in a narrow interval $K \in \left(\frac{c-2}{d}, \frac{c}{d}\right)$.

Figure 2 illustrates the asymptotic stability at the predator-free equilibrium $E_2 = (K, 0)$. To satisfy the stability conditions for E_2 established in the analytical section ($0 < r < 2$ and $dK < c < 2 + dK$), a different parameter set ($r = 1.2$, $b = 0.015$, $c = 1.45$, $d = 0.01$, $K = 120$) was used. The results were obtained through numerical simulation of system of equations (1), with parameters selected to meet these specific analytical conditions.

Figure 2 - Asymptotic stability at $E_2 = (120,\ 0)$.

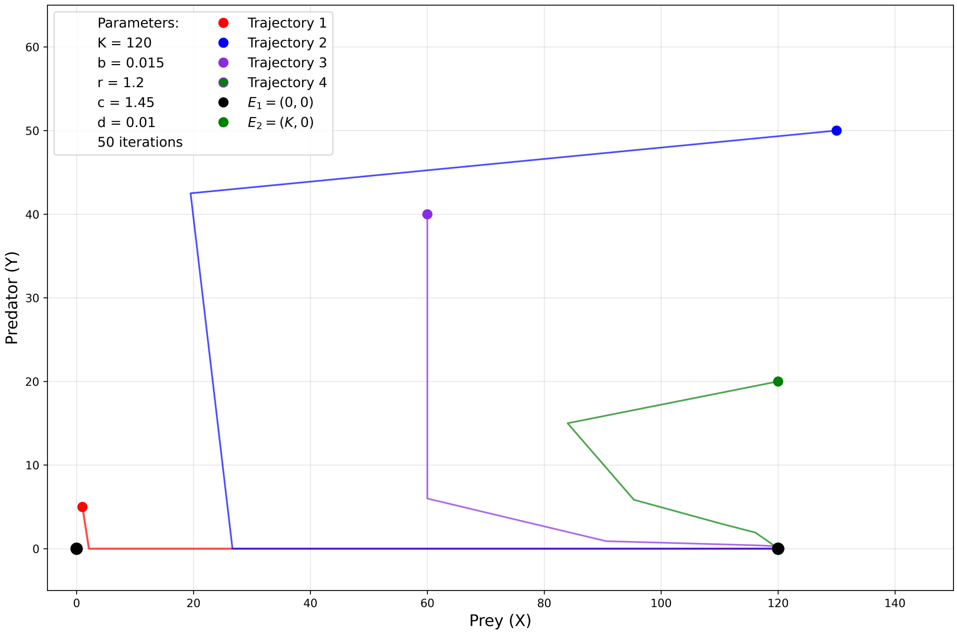


Figure 2 shows that, as Trajectories 1-4 approach sufficiently close to E_2, they begin to converge toward the equilibrium.

Source-type instability

Considering item (b) of Theorem 1 we obtain that E_2 is a source-type unstable equilibrium if,

$$r > 2 \quad \text{and} \quad (c - dK < 0 \quad \text{or} \quad c - dK > 2).$$

Let us consider two relevant scenarios:

- $r > 2$ and $c < dK$: Note that $\lambda_1 < -1$ and $\lambda_2 > 1$. The system trajectory diverges from the equilibrium point.

- $r > 2$ and $c > 2 + dK$: Observe that in this case $\lambda_1, \lambda_2 < -1$, and since $y_n \approx \lambda_2^n y_0$, the predator population trajectory alternates between positive and negative values with increasing amplitudes. While this is mathematically valid behavior within the linearized system context, the ecological interpretation encounters issues due to positivity violation.

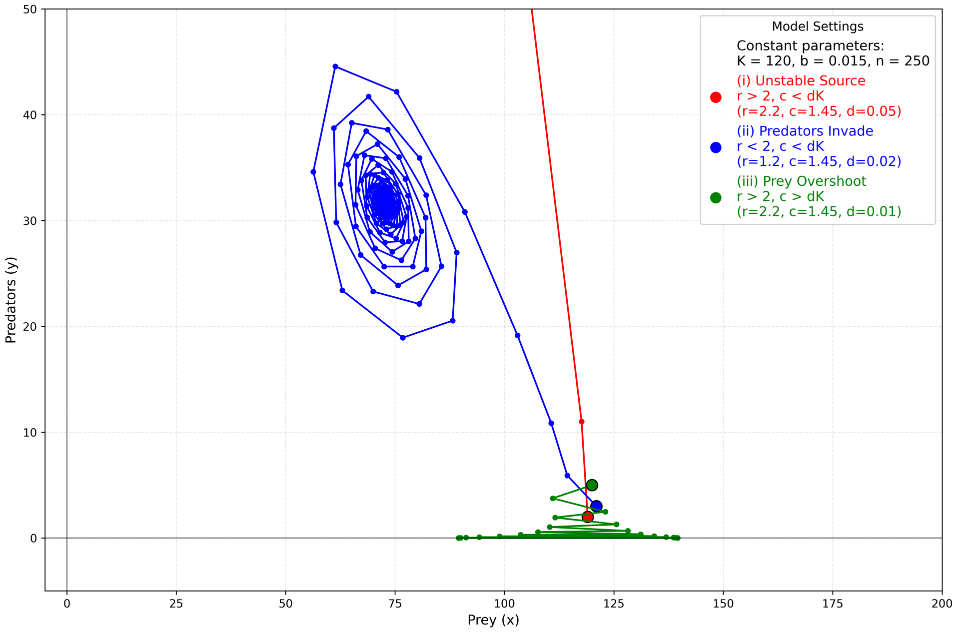
In Figure 3, three systems were configured to demonstrate different types of local instabilities at $E_2 = (120, 0)$, using the parameter values $K = 120$, $b = 0.015$, and $d = 0.01$, and considering 250 iterations:

(i) The unstable source (red trajectory), for which $r > 2$ and $c < dK$ ($r = 2.2, c = 1.45, d = 0.05$);

(ii) Predator invasion (blue trajectory), with $r < 2$ and $c < dK$ ($r = 1.2, c = 1.45, d = 0.02$);

(iii) Prey overshoot (green trajectory), for which $r > 2$ and $c > dK$ ($r = 2.2, c = 1.45, d = 0.01$).

Figure 3 - Three systems configured to illustrate different local instabilities at $E_2 = (120,\ 0)$.



The source-type instability configuration is represented by the red Trajectory 1 in Figure 3, where we observe the progressive divergence from the equilibrium point $E_2 = (K, 0)$.

Saddle Point

We verify that this type of instability manifests in two distinct regimes:

- When $r < 2$ and $c < dK$, we have $|\lambda_1| < 1 < |\lambda_2|$. In this case, we observe moderate prey growth while the predator population increases, characterizing species invasion into the system. This behavior is illustrated by the blue Trajectory 2 in Figure 3.
- When $r > 2$ and $dK < c < 2 + dK$, $|\lambda_2| < 1 < |\lambda_1|$. In this case, we find prey growth exceeding K, causing abrupt variations at each iteration due to the model's nonlinear structure. The predator density, meanwhile, remains near equilibrium without growth tendency, due to the condition $c > dK$. See the green Trajectory 4 in Figure 3 (iii).

Figure 3 illustrates the trajectories of three different systems, obtained through numerical simulation with parameters selected for each instability regime. Each trajectory is initiated near the equilibrium $E_2 = (120, 0)$ and configured to represent the instability regimes discussed previously. The source-type instability is represented by the red Trajectory 1, which shows progressive divergence from the equilibrium point E_2. In Trajectory 2 (blue), we visualize the predator invasion scenario, where the system begins near E_2 and evolves away from this equilibrium, allowing predator population growth. Finally, Trajectory 3 (green) displays the case where prey exceed K, with predators remaining extinct (or near-zero) while prey density shows abrupt variations above and below K at each iteration.

Coexistence equilibrium

This equilibrium $E_3 = (x^*, y^*)$ corresponds to the coexistence of both species in the same environment, with positive populations of prey and predators. To determine its coordinates, we start from system of equations (3), assuming $x^* > 0$ and $y^* > 0$, which allows us to rewrite it as:

$$\begin{cases} 0 &= r\left(1 - \dfrac{x^*}{K}\right) - by^* \\ 0 &= dx^* \; - \; c \end{cases} \tag{12}$$

The second equation of system, equation (12), directly yields

$$x^* = \frac{c}{d}.$$

Substituting this value into the first equation of system, equation (12), we obtain

$$y^* = \frac{r}{Kb}\left(K - \frac{c}{d}\right).$$

Thus, the coexistence equilibrium point is given by

$$E_3 = \left(\frac{c}{d}, \frac{r}{Kb}\left(K - \frac{c}{d}\right)\right), \; K > \frac{c}{d}.$$

Note that E_3 will have both coordinates positive whenever $Kd - c > 0$, guaranteeing biologically viable populations. Proceeding with the local analysis, the Jacobian matrix, equation (9) evaluated at $E_3 = (x^*, y^*)$ takes the form:

$$\mathbf{J}_3^* = \begin{bmatrix} 1 - \dfrac{rc}{dK} & -\dfrac{bc}{d} \\ \dfrac{rd}{bK}\left(K - \dfrac{c}{d}\right) & 1 \end{bmatrix} \tag{13}$$

Although the eigenvalues of the Jacobian matrix, equation (13), can be obtained analytically, the resulting expression, given by

$$\lambda = \frac{2 - \dfrac{rc}{dK} \pm \sqrt{\left(2 - \dfrac{rc}{dK}\right)^2 - 4\left(1 - \dfrac{rc}{dK}(1 + c - dK)\right)}}{2}. \tag{14}$$

is algebraically intricate and not very enlightening. We therefore reserve equation (14) for the analysis of the specific Case II, $\dfrac{rc}{dK} = 2$.

To further analyze the coexistence equilibrium E_3, we now employ the trace-determinant plane approach, although this classical stability criterion appears in various texts under different presentations, we adopt here the formulation in Saber (2007) for its particular convenience in our analysis of the coexistence equilibrium E_3. The trace-determinant criterion proves especially suitable for establishing explicit stability conditions in terms of the trace and determinant of the Jacobian matrix.

To support the previous criteria, we now present a classical result for matrices $A = [a_{ij}]_{2\times2}$, which establishes conditions on the trace $\mathrm{tr}\mathbf{A} = a_{11} + a_{22}$ and determinant $\det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21}$ to guarantee that $\varrho(\mathbf{A}) < 1$. This result, together with Theorem 1, will be instrumental for analyzing the stability of the coexistence equilibrium E_3. For a complete proof and a detailed development of this methodology, see (Saber, 2007), Chapter 4, Theorem 4.4, pages 200–201.

Theorem 2. *(Stability in the Trace-Determinant Plane) Let* $\mathbf{A} = (a_{ij})$ *be a* 2×2 *matrix. Then the spectral radius* $\varrho(\mathbf{A}) < 1$ *if and only if*

$$|\mathrm{tr}\mathbf{A}| - 1 < \det \mathbf{A} < 1. \tag{15}$$

We apply this result to our problem with $\mathbf{A} = \mathbf{J}_3^*$ to identify relationships between the parameters $r,\ b,\ c,\ d$ and K that satisfy inequality (15). For this purpose, we observe that:

$$|\mathrm{tr}\mathbf{J}_3^*| - 1 < \det \mathbf{J}_3^* < 1 \tag{16}$$

Inequality (16) expands to the following explicit form:

$$\left|2 - \frac{rc}{dK}\right| < 2 - \frac{rc}{dK}(1 + c - dK) < 2. \tag{17}$$

Furthermore, the modulus function in inequality (17) can be expanded as

$$\left|2 - \frac{rc}{dK}\right| = \begin{cases} 2 - \dfrac{rc}{dK} & \text{, if } \dfrac{rc}{dK} < 2; \\ 0 & \text{, if } \dfrac{rc}{dK} = 2 \\ \dfrac{rc}{dK} - 2 & \text{, if } \dfrac{rc}{dK} > 2. \end{cases} \tag{18}$$

To resolve inequality (17), we use the piecewise expansion of the modulus function (18) to break the problem into cases. This enables a straightforward application of Theorem 2 to each parameter regime.

Case I: $0 < \dfrac{rc}{dK} < 2$

In this case, inequality (17) is written as:

$$2 - \frac{rc}{dK} < 2 - \frac{rc}{dK}(1 + c - dK) < 2 \tag{19}$$

Developing the expression (19), we obtain

$$\begin{gathered} -\frac{rc}{dK} < -\frac{rc}{dK}(1 + c - dK) < 0 \\ 0 < 1 + c - dK < 1 \\ \frac{c}{d} < K < \frac{c+1}{d}. \end{gathered}$$

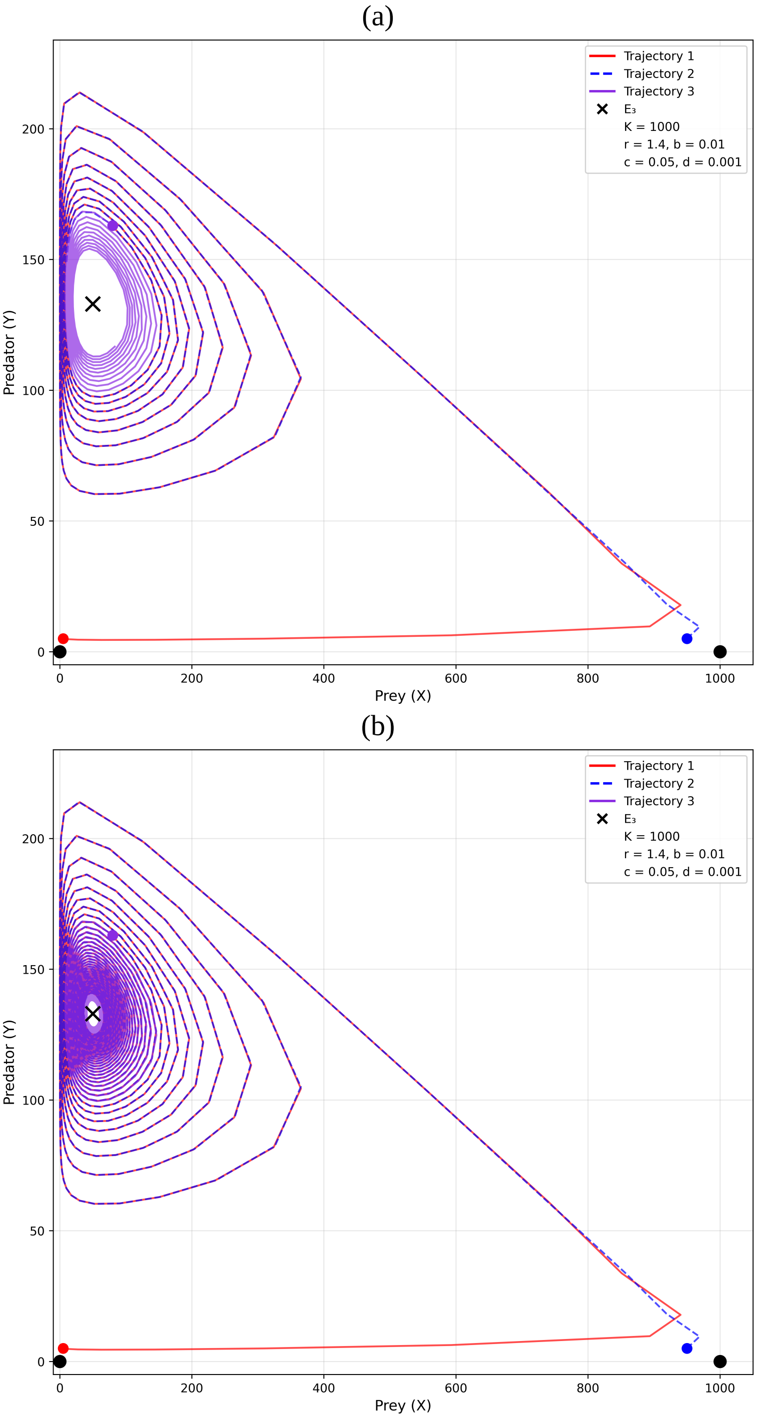
Applying Theorems 1 and 2, we conclude that the equilibrium E_3 is **asymptotically stable** when $K \in \left(\dfrac{c}{d}, \dfrac{c+1}{d}\right)$.

The condition $K \in \left(\dfrac{c}{d}, \dfrac{c+1}{d}\right)$ is equivalent to $dK - 1 < c < dK$, and in biological terms, this means the mortality rate c cannot be excessively low (which would lead to instability) nor excessively high (which would prevent predator persistence). This result highlights the sensitivity of the system's stability to small variations in mortality rate, particularly relative to maximum predation (dK).

To illustrate the local asymptotic stability of E_3, we return to the parameter set used in Figure 1 ($r = 1.4$, $b = 0.01$, $c = 0.05$, $d = 0.001$), which satisfies the asymptotic stability conditions established for E_3. Figure 4 shows numerical simulations of system of equation (1), approach to this equilibrium from three different initial conditions: Trajectory 1 (red) starts near the trivial equilibrium E_1; Trajectory 2 (blue) starts near the predator-free equilibrium E_2; and Trajectory 3 (purple) starts near the coexistence equilibrium E_3.

Figure 4 illustrates how the dynamics approach equilibrium, using the parameter values $r = 1.4$, $b = 0.01$, $c = 0.05$, and $d = 0.001$, showing 250 and 1000 iterations, respectively.

Figure 4 - Asymptotically stable at E_3, with the same parameters as in Figure 1: (a) 250 iterations and (b) 1000 iterations.



Case II: $\dfrac{rc}{dK} = 2$

Since we have an explicit value, we will use a combination of two tools: eigenvalue computation from (14) and application of Theorem 1. Thus, the eigenvalues are:

$$\lambda_{1,2} = \pm\sqrt{1 + 2c - 2dK}. \tag{20}$$

Stability requires $|\lambda_{1,2}| < 1$ (Theorem 1), which is equivalent to $|\lambda_{1,2}|^2 < 1$. To eliminate the square root in equation (20), we obtain:

$$|\lambda_{1,2}|^2 = |1 + 2c - 2dK|. \tag{21}$$

Substituting equation (21) into the stability criterion $|\lambda_{1,2}|^2 < 1$ gives:

$$\begin{aligned} |\lambda_{1,2}|^2 < 1 &\iff |1 + 2c - 2dK| < 1 \\ &\iff -1 < 1 + 2c - 2dK < 1 \\ &\iff 0 < dK - c < 1 \end{aligned}$$

If $(dK - c) \in (0,\,1) \Rightarrow |\lambda|^2 < 1 \Rightarrow |\lambda| < 1$.

Note that this condition mirrors Case I, highlighting the same ecological constraints.

Remark: For $\dfrac{rc}{dK} > 2$, algebraic manipulations do not preserve the equivalence of inequality (17), yielding only partial implications. Investigating this interval requires different methods, whose analysis will be considered in future work.

The local stability analysis revealed explicit conditions for each system. Table 1 summarizes these conditions as functions of biological parameters, highlighting the associated equilibrium regimes.

Table 1 - Classification of the system's equilibria according to parameter values.

Equilibrium	**Parameters**		**Classification**
$E_1 = (0,0)$		$c > 2$	Unstable source
		$0 < c < 2$	Unstable saddle point
$E_2 = (K,0)$	$0 < r < 2$	$dK < c < 2 + dK$	Asymptotically Stable
		$c < dK$ or $c > 2 + dK$	Unstable saddle point
	$r > 2$	$dK < c < 2 + dK$	
		$c < dK$ or $c > 2 + dK$	Unstable source
$E_3 = \left(\dfrac{c}{d}, \dfrac{r}{Kb}\left(K - \dfrac{c}{d}\right)\right)$	$0 < \dfrac{rc}{dK} < 2$ $\dfrac{rc}{dK} = 2$	$K \in \left(\dfrac{c}{d}, \dfrac{c+1}{d}\right)$	Asymptotically Stable

Conclusions

This work has presented a local stability analysis for a system of the Lotka-Volterra type formulated directly in discrete time. The main contribution lies in obtaining explicit and interpretable stability conditions for each of the three biologically relevant equilibria, based on linearization and spectral analysis of the Jacobian matrix.

Our stability analysis reveals three distinct dynamical regimes governed by the system's biological parameters. This classification enables comparative identification of critical intervals for coexistence, exclusion, or population collapse, serving as a practical reference for both theoretical interpretations and future applications in simulations or management. The analysis establishes that:

- The total extinction equilibrium (E_1) is always unstable. This result is biologically consistent: since the prey growth rate $r > 0$ and predators are near zero, any small initial prey population will inevitably grow, leading the system away from the trivial equilibrium.
- The asymptotic stability of the predator-free equilibrium (E_2) requires the predator mortality rate (c) to exceed the benefit gained from predation when prey are at carrying capacity (dK) - a condition that

validates the model, as ecologically predators are not expected to invade a system where mortality exceeds the gains from prey availability. Simultaneously, it requires the prey growth rate to remain between 0 and 2, reaffirming the stability boundary established by May (1976) for the discrete logistic equation that models prey growth.

- The asymptotic stability of coexistence (E_3) reveals a high sensitivity to parameter variations, occurring only within narrow ranges of the carrying capacity K, determined by the predator mortality-to-regeneration ratio ($\frac{c}{d} < K < \frac{(c+1)}{d}$). The requirement for fine parameter tuning ($0 < \frac{rc}{dK} \leq 2$) reinforces that small variations can destabilize the coexistence equilibrium, leading to predator extinction or population collapse. Furthermore, parameter rearrangement reveals that dK must lie within the interval $(c, c+1)$ - a fixed interval of width 1. This implies that the net benefit from predation (dK) must be quite close to the mortality rate (c) for coexistence to be stable, demonstrating significant parametric sensitivity.

Although the parameter b, representing the predation rate, does not directly affect local stability criteria, it determines the predator density at the coexistence equilibrium. Thus, its variation strongly influences the configuration of the coexistence equilibrium or ecological viability.

Beyond the theoretical results, this work provides a pedagogical synthesis and application of classical stability methods for discrete systems. This unified framework, which includes linearization, spectral analysis, and the trace-determinant criterion, is computationally efficient as it is developed entirely in discrete time. The numerical simulations, while challenging to calibrate, were designed not only to visualize the dynamic regimes but also to show the practical utility of this methodological integration in an ecological context. This approach, which explicitly bridges analytical results with interpretable dynamics, establishes discrete modeling as a powerful tool for both education and the analysis of ecosystems with sparse monitoring data, providing a foundation for future studies on management strategies.

Finally, although the model depicts natural interactions, its transposition to the human realm reveals a reality difficult to ignore: the same species capable of rapid regeneration and longevity exerts, through its actions, one of the most intense predatory pressures on the very system that sustains it. Recognizing this force is not an invitation to resignation, but to active awareness. Models like this not only outline the fragile contours of stability but also remind us that it endures only when understood and respected.

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Author Contributions

F. F. C. Puma: conceptualization, project managements, formal analysis, data curation, investigation, methodology, supervision, validation, visualization and writing – preparation of the original draft , revision and editing. **M. M. Mendiguren**: formal analysis, visualization, supervision, validation and writing – preparation of the original draft , revision and editing.

Conflicts of Interest

The authors declare no conflict of interest.

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