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Population Growth Models Using Fuzzy Ordinary Differential Equations

Modelos de Crescimento Populacional através de Equações Diferenciais Ordinárias Fuzzy

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ABSTRACT

Fuzzy mathematics is a branch of mathematics that presents a new approach to the classical notion of set, enabling the generalization of concepts and results from classical mathematics. Using the concept of fuzzy number, a generalization of the real number, we study fuzzy ordinary differential equations, generating ampler solutions to differential equations that model population growth. In particular, we examined a model for the growth of microorganisms in milk. By applying the fuzzy solution we expanded the possible parameter values of the models and studied their relationships. In this context, we study the properties of interactive fuzzy numbers, as well as their applications in mathematical modeling. Through concrete examples, we illustrate the application of fuzzy mathematics and interactive fuzzy numbers in real-world situations, highlighting their relevance in decision-making in contexts of uncertainty and imprecision. Leveraging the solutions and parameters values reported in the literature, the theory of interactive fuzzy numbers was used to develop predictive models. These models encompass intervals centered around the values deterministically predicted values, incorporating deviations within the confidence interval obtained for one of the parameters in each model.

keywords fuzzy mathematics, population growth models, ordinary differential equations, fuzzy numbers, mathematical modelling

RESUMO

A matemática fuzzy é uma área da matemática que apresenta uma nova abordagem para a noção clássica de conjunto, permitindo a generalização de conceitos e resultados da matemática clássica. Usando o conceito de número fuzzy, uma generalização do número real, estudamos equações diferenciais ordinárias fuzzy, gerando soluções mais amplas para equações diferenciais que modelam o crescimento populacional. Em particular, estudamos um modelo para o crescimento de microrganismos no leite e, por meio da solução fuzzy, expandimos os possíveis valores de parâmetros dos modelos e estudamos suas relações. Nesse contexto, estudamos as propriedades dos números fuzzy interativos, bem como suas aplicações na modelagem matemática. Por meio de exemplos concretos, ilustraremos como a Matemática Fuzzy e os números fuzzy interativos são aplicados em situações do mundo real, destacando sua relevância na tomada de decisões em ambientes de incerteza e imprecisão. Através das soluções e valores dos parâmetros já obtidos na literatura, juntamente com teoria dos números fuzzy interativos, obtivemos modelos preditivos que abarcam intervalos centrados nos valores previstos deterministicamente, através de desvios previstos dentro do intervalo de confiança obtido para um dos parâmetros em cada modelo.

palavras-chave matemática fuzzy, modelos de crescimento populacional, equações diferenciais ordinárias, números fuzzy, modelagem matemática

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Introduction

A classic approach to modeling the behavior of populations over time is to represent the population as a real function of a time-dependent variable, and using information about its variation over time, study Ordinary Differential Equations (ODEs) that represent the population (Bassanezi, 2002; Gomes et al., 2015).

However, when comparing the models resulting from ODEs as we know them, they are characterized as deterministic models, i.e. given an instant t, the model indicates exactly one possible value for the population. However, when we empirically test the models, the result is only data that is close to the prediction.

We propose the study of Fuzzy Ordinary Differential Equations, i.e. ODEs in which the solution is a fuzzy function $f(t)$, associating to each real number t, a fuzzy number as its image. In this context, we will study the properties of fuzzy numbers and, especially, the class of interactive fuzzy numbers, as well as their applications in mathematical modeling, in particular in microbial population growth models already established in the literature, known as Logistic modified model (LMZ), and Gompertz modified model (GOM) (Longhi et al., 2013, 2017).

Fuzzy mathematics is a field of study that reinterprets the notion of set, which, being extremely basic to modern mathematical activity, allows for the generalization of concepts and results from classical mathematics. This theory was introduced in the second half of the 20th century, in the work of (Zadeh, 1965), through the fundamental idea of fuzzy sets and covers subsequent notions such as support, operations with fuzzy sets and, at the center of our interests: fuzzy numbers.

A simple method for extending classical solutions to fuzzy solutions, i.e. a fuzzy solution that includes the classical solution, is to apply Zadeh's extension principle, which we present in later sections. Naturally, we also introduce the concept of fuzzy numbers. As a result, given an ODE and its solution, we can obtain a fuzzy function that includes the solution of the original ODE.

Our aim is to extend the LMZ and GOM microbial growth models using Zadeh's extension principle, taking one parameter of the solution as a fuzzy number. As we present later, the parameters $\mu_{\max}$ and λ are given by a hyperbolic function. This implies that, when λ is given as a fuzzy number, then $\mu_{\max}$ a fuzzy number determined by the values of λ and the hyperbolic function, that means $\mu_{\max}$ and λ are examples of f-correlated fuzzy numbers (Cabral et al., 2015).

Firstly, we present the basis for studying microbial growth using ODEs, mainly the LMZ and GOM models, and their particularities. Next, we present some concepts and results from fuzzy mathematics, such as fuzzy numbers, Zadeh's extension principle and interactivity between fuzzy numbers. This mathematical background constitutes our material and methods. As results, we present fuzzy functions that extend the solutions of the LMZ and GOM models, considering the parameters $\mu_{\max}$ and λ as fuzzy numbers estimating possible deviations from the classical solutions through possible deviations of these parameters.

Materials and methods

Now we discuss the LMZ and GOM models, given by ODEs, and also their parameters. We introduce some concepts of fuzzy mathematics, such as fuzzy numbers, especially f-correlated fuzzy numbers. This enables us to extend the classical solutions obtained for LMZ and GOM using fuzzy numbers that generalize the classical values of the parameters.

In predictive microbiology, some of the most used models are the Gompertz and Logistic (Robazza et al., 2010). Their mathematical equations and respective solutions are already established in the literature. We present both in Table 1, where P is the population in function of time t, P_0 is the initial population, P_∞ is the population asymptote, a, b and k are real numbers that indicate the proportionality between population and its variation $\frac{dP}{dt}$.

The LMZ and GOM are examples of sigmoidal models, having phases, namely: the lag phase, when the population is adapting to the environment, the exponential phase, when the population is growing and is already adapted, the stationary phase, when the population finds a balance between deaths and births, caused by limited resources in the environment, and the decline phase, when the number of deaths increasingly exceeds the number of new individuals (Robazza et al., 2010).

Table 1 - Solutions for the classical Logistic and Gompertz models

Model	Ordinary Differential Equation	Solution
Logistic	$\begin{cases} \dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{P_\infty}\right) \\ P(0) = P_0 \end{cases}$	$P(t) = P_\infty \left[1 + \left(\dfrac{P_\infty}{P_0} - 1\right) e^{-kt}\right]^{-1}$
Gompertz	$\begin{cases} \dfrac{dP}{dt} = P(a - b\ln P) \\ P(0) = P_0 \end{cases}$	$P(t) = P_\infty \left(\dfrac{P_0}{P_\infty}\right)^{e^{-bt}}$

From "Ensino-aprendizagem com modelagem matemática" by R. Bassanezi, 2002.

When we consider the values of the constants a, b and k in function of biologically significant parameters, we obtain reparameterized models, examples of which are LMZ and GOM, presented in Table 2. The variable t represents time in hours (h). The function $Y(t)$ is defined as the logarithmic ratio between the bacterial count N at time t and the initial bacterial count N_0, expressed as $Y(t) = \ln[N(t)/N_0]$. The parameter μ_{max} denotes the maximum specific growth rate (1/h), while λ represents the duration of the lag phase (h). The value k corresponds to the maximum slope of the growth curve. The amplitude of the curve, A, is given by the logarithmic ratio between the maximum bacterial count N_{max} and the initial count N_0, calculated as $A = \ln(N_{\text{max}}/N_0)$ (Longhi et al., 2017).

Table 2 - Solutions for the modified Logistic and Gompertz models

Model	Ordinary Differential Equation	Solution
Modified Logistic (LMZ)	$\begin{cases} \dfrac{dY(t)}{dt} = \mu_{\max}\left(\dfrac{4}{A}\right)\left[1 - \left(\dfrac{Y(t)}{A}\right)\right] Y(t) \\ Y_0 = \dfrac{A}{1 + e^{\lambda\mu_{\max}\left(\frac{4}{A}\right)+2}} \end{cases}$	$Y(t) = \dfrac{A}{1 + e^{\mu_{\max}\left(\frac{4}{A}\right)(\lambda - t)+2}}$
Modified Gompertz (GOM)	$\begin{cases} \dfrac{dY(t)}{dt} = \mu_{\max}\left(\dfrac{e}{A}\right)\left[\ln\left(\dfrac{A}{Y(t)}\right)\right] Y(t) \\ Y_0 = Ae^{-e^{\frac{\lambda\mu_{\max}e}{A}+1}} \end{cases}$	$Y(t) = Ae^{-e^{\left(\mu_{\max}\frac{e}{A}\right)(\lambda - t)+1}}$

From “Microbial growth models: A general mathematical approach to obtain μ_{max} and λ parameters from sigmoidal empirical primary models”, by D. A. Longhi, F. Dalcanton, G. M. F. Aragão, B. A. M. Carciofi, and J. B. Laurindo (2017), *Brazilian Journal of Chemical Engineering*, 34(2), 369–375.

It is important to note that the parameters λ and $\mu_{\max}$ can be written in a hyperbolic function. That means, given the inflection point t_{ifx}, the natural logarithm of the microbial count at the moment of inflection y_{ifx} and at the initial moment y_0, the parameter λ varies depending on the parameter $\mu_{\max}$, according to a hyperbolic function, i.e. of the form $f(x) = \frac{q}{x} + r$ (Longhi et al., 2013), as we can see in equation (1):

$$\lambda = t_{ifx} - \frac{y_{ifx} - y_0}{\mu_{\max}}. \tag{1}$$

The time at the inflection point (t_{ifx}) is the root in the equation given by equations (2)

$$\frac{d^2y(t)}{dt^2} = 0, \tag{2}$$

and the model response at the inflection point (y_{ifx}) is obtained by substituting t_{ifx} in the original model equation, as shown in equations (3)

$$y(t_{ifx}) = y_{ifx}, \tag{3}$$

while the maximum specific growth rate (μ_{max}) is obtained by substituting t_{ifx} into the first derivative of the sigmoid model, as shown in equations (4)

$$\frac{dy(t_{ifx})}{dt} = \mu_{\text{max}}. \tag{4}$$

In our study, equation (1) implies that if $\mu_{\max}$ is a fuzzy number, then λ is also a fuzzy number, hyperbolically correlated to $\mu_{\max}$. We now present these concepts.

Fuzzy mathematics

Fuzzy mathematics emerged in the second half of the 20th century, through the notion of a fuzzy set (citeZadeh. The idea is to generalize the concept of set, in particular the dichotomous notion of membership or non-membership, through the concepts of degrees of membership. The following definitions, unless otherwise stated, can be consulted in the book "A First Course in Fuzzy Logic, Fuzzy Dynamical Systems, and Biomathematics Theory and Applications" (Barros et al., 2017).

Therefore, a fuzzy set is identified by a function in the form $\varphi_A U \rightarrow [0, 1]$, while a classical set is represented by its characteristic function, i.e. a function $\chi_B : U \rightarrow \{0, 1\}$ so that $B = \{x \in U; \chi_B(x) = 1\}$, being a particular case of a fuzzy set. Here, the domain U is a universal set and the output of the function represents the degree of membership: 0 represents no association, progressively increasing to 1, representing total association. In our study, our universal set is the real line $\mathbf{R}$.

Lemma 1 Let A and B be fuzzy subsets of a topological space U. A necessary and sufficient condition for $A = B$ to hold is that $[A]^\alpha = [B]^\alpha$ for all $\alpha \in [0, 1]$.

An important concept introduced with the notion of fuzzy set is the Zadeh extension, which allows to "transform" a classical function, i.e., a function in the traditional sense, into a function between a classical set and a fuzzy set.

Definition 1 (Zadeh, 1965) Let f be a function such that $f : X \longrightarrow Z$ and A be a fuzzy subset of X. The Zadeh extension of f is the function $\hat{f}$ which, when applied to A, gives the fuzzy subset $\hat{f}(A)$ of Z, whose membership function is given by the system presented in equation (5):

$$\varphi_{\hat{f}(A)}(z) = \begin{cases} \sup\limits_{f^{-1}(z)} \varphi_A(x) & \text{if } f^{-1}(z) \neq \emptyset \\ 0 & \text{if } f^{-1}(z) = \emptyset \end{cases} \tag{5}$$

where $f^{-1}(z) = \{x; f(x) = z\}$ denotes the preimage of z.

Zadeh introduced not only a generalization of set and a way to extend classical functions, but also extended the operations with sets, namely union, intersection, and complement. Of course, we suppose fixed our universal set U.

Definition 2 (Zadeh, 1965) Given A and B fuzzy sets, the union of A and B is the fuzzy set given by $\varphi_{A\cup B}(x) = \min\{\varphi_A(x), \varphi_B(x)\}$, their intersection is the fuzzy set given by $\varphi_{A\cap B}(x) = \max\{\varphi_A(x), \varphi_B(x)\}$, and the complement of A is the fuzzy set given by $\varphi_{A^C}(x) = 1 - \varphi_A(x)$.

The functions $\min$ and $\max$ can be generalized through the concepts of t-norm and t-conorm. The t-norm can introduce a class of interactive fuzzy numbers and also generalize the Zadeh extension principle.

Definition 3 (Cabral & Barros, 2019) The operator $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a triangular norm (or simply t-norm) if for all $x, y, z, w \in [0, 1]$, the properties below are satisfied:

t1) $T(x, y) = T(y, x)$ (Commutative)

t2) $T(x, T(y, z)) = T(T(x, y), z)$ (Associative)

t3) If $y \leq z$ then $T(x, y) \leq T(x, z)$ (Monotonicity)

t4) $T(x, 1) = x$ (Neutral Element).

The following four functions are examples of t-norms: t-norm of minimum T_M, of the product T_P, of Lukasiewicz T_L and drastic product T_D, which are defined respectively by equations (6)-(9):

$$T_M(x,y) = \min\{x,y\}; \tag{6}$$

$$T_P(x,y) = x \cdot y; \tag{7}$$

$$T_L(x,y) = \max\{x+y-1,0\}; \tag{8}$$

$$T_D(x,y) = \begin{cases} 0, & \text{if } x \neq 1 \text{ and } y \neq 1 \\ x, & \text{if } y = 1 \\ y, & \text{if } x = 1 \end{cases}. \tag{9}$$

Definition 4 The operator $S : [0,1] \times [0,1] \to [0,1]$ is called a triangular conorm (or simply t-conorm) if for all $x, y, z, w \in [0,1]$, the properties below are satisfied:

t1) $S(x,y) = S(y,x)$ (Commutative)

t2) $S(x, S(y,z)) = S(S(x,y), z)$ (Associative)

t3) If $y \leq z$ then $S(x,y) \leq S(x,z)$ (Monotonicity)

t4) $S(x,0) = x$ (Neutral Element).

The following three functions are examples of t-conorms: t-conorm of maximum S_M, t-conorm of algebraic sum S_s, and t-conorm of Lukasiewicz S_L, defined respectively by equations (10)-(12):

$$S_M(x,y) = \max\{x,y\} = x \vee y; \tag{10}$$

$$S_s(x,y) = x + y - xy; \tag{11}$$

$$S_L(x,y) = \min\{1, x+y\}. \tag{12}$$

Definition 5 A function $N : [0,1] \longrightarrow [0,1]$ is a negation if it satisfies the following conditions:

n_1 Boundary conditions: $N(0) = 1$ and $N(1) = 0$;

n_2 Monotonicity: N is decreasing;

moreover if N is strictly decreasing, (n_1) holds and

n_3 Involution: $N(N(x)) = x$,

then N is called *strong negation*.

The functions $N_1(x)$ and $N_2(x)$, given in equations (13) and (14)

$$N_1(x) = 1 - x, \tag{13}$$

and

$$N_2(x) = \frac{1-x}{1+x}, \tag{14}$$

are strong negations. However, $N_3(x)$, equation (15)

$$N_3(x) = \begin{cases} 0 & \text{if } x = 1; \\ 1 & \text{if } x \in [0,1[, \end{cases} \tag{15}$$

is a negation but not a strong one.

We say that the t-norm T and the t-conorm S are *dual* with respect to a negation N if they satisfy the two laws of De Morgan. We note that the operations T_M, S_M, and N_1, defined by equations (6), (10) and (13), respectively, satisfy the De Morgan's laws, that is, for all pairs (x, y) of $[0, 1] \times [0, 1]$, as expressed in equations (16) and (17)

$$N_1(T_M(x, y)) = S_M(N_1(x), N_1(y)); \quad (16)$$

$$N_1(S_M(x, y)) = T_M(N_1(x), N_1(y)). \quad (17)$$

More specifically, we have the following result.

Theorem 1 Operations between fuzzy subsets have the following properties, for A, B and C fuzzy subsets of U:

a. $A \cup B = B \cup A$;

b. $A \cap B = B \cap A$;

c. $A \cup (B \cup C) = (A \cup B) \cup C$;

d. $A \cap (B \cap C) = (A \cap B) \cap C$;

e. $A \cup A = A$;

f. $A \cap A = A$;

g. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

h. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;

i. $A \cap \emptyset = \emptyset$ and $A \cup \emptyset = A$;

j. $A \cap U = A$ e $A \cup U = U$;

k. $(A \cap B)' = A' \cup B'$

l. $(A \cup B)' = A' \cap B'$;

An even more general version of the extension principle can be defined through the notion of possibility distributions, along with the general notion of interactivity of fuzzy numbers.

Definition 6 (Dubois & Prade, 1981) A possibility distribution over $\Omega \neq \emptyset$ is a function $\varphi : \Omega \to [0, 1]$ that satisfies $\sup_{\omega \in \Omega} \varphi(\omega) = 1$.

For our next definitions, we will need to know what exactly is a fuzzy number. Then, we can define interactivity between fuzzy numbers, and we highlight two particular cases: interactivity via a t-norm and f-correlation.

To define fuzzy numbers, we have to define the α levels of the fuzzy sets. Given $\alpha \in (0, 1]$, α-level of A is $[A]^\alpha = \{a \in U; \varphi_A \geq \alpha\}$. The support of A is supp$A = \{a \in U; \varphi_A > 0\}$. And for $\alpha = 0$, the α-level of A is $[A]^\alpha = \overline{\text{supp}A}$.

Definition 7 A fuzzy subset A of $\mathbf{R}$ is a fuzzy number if every α-level is a non-empty bounded closed interval of $\mathbf{R}$ with closure of the support.

Example 1 Every real number x is a fuzzy number, with $[x]^\alpha = [x, x]$, for all $\alpha \in [0, 1]$. More generally, every interval $X = [a, b] \subset \mathbf{R}$ is a fuzzy number with $[X]^\alpha = [a, b]$, for all $\alpha \in [0, 1]$.

We denote the set of fuzzy numbers by $\mathcal{F}_{\mathbf{R}}$ and the set of joint possibility distributions over $\mathbf{R}^n$ by $\mathcal{F}_C(\mathbf{R}^n)$. In our study, we present extensions of real functions of one variable, more specifically, associating a real number to a fuzzy number. In our next definitions, we will need some concepts of possibility theory.

Definition 8 (Dubois & Prade, 1981; Zadeh, 1975) Let A and B be fuzzy numbers and $C \in \mathcal{F}_C(\mathbf{R}^2)$. Then, φ_C is a joint possibility distribution of A and B if the relations in equation (18) holds:

$$\max_{y \in \mathbf{R}} \varphi_C(x, y) = \varphi_A(x) \text{ and } \max_{x \in \mathbf{R}} \varphi_C(x, y) = \varphi_B(y). \tag{18}$$

Moreover, φ_A and φ_B are called the marginal distributions of C.

Definition 9 (Carlsson et al., 2004) Let C be the joint possibility distribution of (marginal possibility distributions) $A_1, \ldots, A_n \in \mathcal{F}_{\mathbf{R}}$, and let $f : \mathbf{R}^n \to \mathbf{R}$ be a continuous function. Then $f_C(A_1, \ldots, A_n)$ will be defined by equation (19):

$$f_C(A_1, \ldots, A_n)(y) = \sup_{y = f(x_1, \ldots, x_n)} C(x_1, \ldots, x_n). \tag{19}$$

Definition 10 (Carlsson et al., 2004; Dubois & Prade, 1981; Zadeh, 1975) Fuzzy numbers $A_1, \ldots, A_n$ are said to be *non-interactive* if, for all $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $\alpha \in [0, 1]$, their joint possibility distribution C satisfies the relationship

$$C(x_1, \ldots, x_n) = \min\{A_1(x_1), \ldots, A_n(x_n)\},$$

or, equivalently the relation given by the equation (20)

$$[C]^\alpha = [A_1]^\alpha \times \cdots \times [A_n]^\alpha. \tag{20}$$

Otherwise, they are said to be interactive.

Lemma 2 (Carlsson et al., 2004) Let $A_1, \ldots, A_n$ be fuzzy numbers, let C be their joint possibility distribution, and let $f : \mathbf{R}^n \to \mathbf{R}$ be a continuous function. Then, for all $\alpha \in [0, 1]$, equation (21) holds:

$$[f_C(A_1, \ldots, A_n)]^\alpha = f([C]^\alpha). \tag{21}$$

This definition generalizes the next concepts. We highlight two types of interactivity: via t-norm, and specially, f-correlation, which is the case of the parameters we study in the LMZ and GOM models.

Definition 11 (Cabral & Barros, 2019) Two fuzzy subsets $A \in \mathbf{R}^n$ and $B \in \mathbf{R}^m$ are said to be interactive via t-norm T if the joint possibility distribution C_T of A and B is defined by a t-norm T, that is, $\mu_{C_T} : \mathbf{R}^n \times \mathbf{R}^m \to [0, 1]$ is such that $\mu_{C_T}(x, y) = T(\mu_A(x), \mu_B(y))$.

Definition 12 (Cabral & Barros, 2019) Let $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^k$ be a function and $A \in \mathcal{F}(\mathbf{R}^n)$ and $B \in \mathcal{F}(\mathbf{R}^m)$ be interactive fuzzy subsets via t-norm T. The extension of f applied to (A, B) via T is the fuzzy subset $f_T(A, B)$ of $\mathbf{R}^k$ whose membership function is defined by the system given in equation (22):

$$\mu_{f_T(A,B)}(w) = \begin{cases} \sup\limits_{(u,v) \in f^{-1}(w)} T(\mu_A(u), \mu_B(v)), & \text{if } f^{-1}(w) \neq \emptyset, \\ 0, & \text{if } f^{-1}(w) = \emptyset, \end{cases} \tag{22}$$

where $f^{-1}(w) = \{(u, v) : f(u, v) = w\}$.

It is important to note that **Definition 12** is in fact, an example of Definition 9 when the joint possibility distribution of A and B is given by a t-norm T. However, another important case of interactivity between fuzzy numbers is f-correlation, and it does not depend on a t-norm.

Definition 13 (Cabral et al., 2015) Let $X, Y \subset \mathbf{R}$ and $f : X \to Y$ be a monotonic, injective, and continuous function. Two fuzzy numbers A and B are correlated according to the function f or f-correlated if their joint possibility distribution J is given by equation (23)

$$\varphi_J(x,y) = \varphi_A(x)\chi_{y=f(x)}(x,y) = \varphi_B(y)\chi_{y=f(x)}(x,y), \tag{23}$$

where $\chi_{y=f(x)}(x,y)$ is the characteristic function of the graph of f.

Example 2 (Carlsson et al., 2004) Two fuzzy numbers A and B are declared completely correlated if there exists $q, r \in \mathbf{R}$, with $q \neq 0$, such that their joint possibility distribution, with $\chi_{qx+r=y}(x,y)$ being the characteristic function of the line $\left\{(x,y) \in \mathbf{R}^2; qx + r = y\right\}$, is defined by equation (24):

$$\varphi_C(x,y) = \varphi_A(x)\chi_{qx+r=y}(x,y) = \varphi_B(y)\chi_{qx+r=y}(x,y).$$

Example 3 Consider f being a hyperbolic function $f : \mathbf{R} \longrightarrow \mathbf{R}$, given by $f(x) = \dfrac{q}{x} + r, x \neq 0$. Two fuzzy numbers A and B are hyperbolically interactive if there exist $q, r \in \mathbf{R}$, $q \neq 0$, such that their joint possibility distribution J is given by

$$\varphi_J(x,y) = \varphi_A(x)\chi_{\frac{q}{x}+r=y}(x,y) = \varphi_B(y)\chi_{\frac{q}{x}+r=y}(x,y),$$

where $\chi_{\frac{q}{x}+r=y}(x,y)$ is the characteristic function of the set $\left\{(x,y) \in \mathbf{R}^2; \frac{q}{x} + r = y\right\}$. In this case, if $0 \notin [A]^0$, then $[B]^\alpha = \left\{\frac{q}{x} + r; x \in [A]^\alpha\right\}$.

Results and discussion

General interactive fuzzy solutions

Combining equation (1) with our knowledge about fuzzy numbers, when we consider $\mu_{\max}$ as a fuzzy number, we obtain:

$$[\lambda]^\alpha = t_{ifx} - \frac{y_{ifx} - y_0}{[\mu_{\max}]^\alpha}. \tag{24}$$

If we suppose $\mu_{\max}$, in equation (24), is a triangular fuzzy number, i.e. there are a and b real numbers such that for all $\alpha \in [0,1]$,

$$[\mu_{\max}]^\alpha = [\alpha(u-a) + a, \alpha(u-b) + b]. \tag{25}$$

Substituting equation (25) into the equation (24), we obtain:

$$[\lambda]^\alpha = t_{ifx} - \frac{y_{ifx} - y_0}{[\alpha(u-a) + a, \alpha(u-b) + b]}. \tag{26}$$

Now, we finally obtain the joint possibility distribution J of the fuzzy numbers λ and $\mu_{\max}$:

$$\varphi_J(x,y) = \varphi_\lambda(x)\chi_{\left\{t_{ifx} - \frac{y_{ifx} - y_0}{x} = y\right\}}(x,y). \tag{27}$$

For simplicity, we will denote $B = t_{ifx}$ and $C = y_{ifx} - y_0$. Thus, we can write:

$$[\lambda]^\alpha = B - \frac{C}{[\mu_{\max}]^\alpha}. \tag{28}$$

We will now consider the Logistic Model, but with λ and $\mu_{\max}$ as fuzzy numbers. Its *fuzzy interactive solution* is the solution where we use the previous equality to work with just one parameter. Therefore, taking the values of λ according to equation (28) and applying **Lemma 2**, we can obtain the fuzzy interactive solution of the Modified Logistic Model, given by:

$$Y(t) = \frac{A}{\exp\left[\mu_{\max}\left(\frac{4}{A}\right)(B - \frac{C}{\mu_{\max}} - t) + 2\right] + 1}. \tag{29}$$

Similarly, the fuzzy interactive solution of the Modified Gompertz Model is given by:

$$Y(t) = A\exp\left\{-\exp\left[\left(\mu_{\max}\frac{e}{A}\right) + (B - \frac{C}{\mu_{\max}} - t) + 1\right]\right\}. \tag{30}$$

In any case, we can write the solution in the following form:

$$[Y(t)_J\,(\mu_{\max}, \lambda)]^{\alpha} = \{y_t(\mu, \lambda); \mu, \lambda \in [J]^{\alpha}\}. \tag{31}$$

Solutions for 20º C

The actual values of the parameters are influenced by diverse factors, specially temperature. The LMZ and GOM models were empirically tested in isothermal and non-isothermal models, and we now present some values obtained for A, $\mu_{\max}$ and λ for the temperatures of 20º C (Longhi et al., 2013). These values are presented in Table 3.

Table 3 - Values of for parameters$A$, $\mu_{\max}$ and λ for the temperatures of 20º C.

	LMZ	**GOM**
A	2.35	2.42
λ	3.71	3.15
$\mu_{\max}$	0.1814	0.1766

From "Assessing the prediction ability of different mathematical models for the growth of lactobacillus plantarum under non-isothermal conditions", by D. A. Longhi, F. Dalcanton, G. M. F. Aragão, B. A. M. Carciofi and J. B. Laurindo, 2013, *Journal of Theoretical Biology*, 335, 88–96.

To determine the values of A, B and C in equations (29) and (30), we need to solve equation (2) for each model, i.e. we have to obtain the inflection points of the solutions in Table 2. Since we are only extending the parameters λ and $\mu_{\max}$,

For the LMZ model, we define $K_1 = \frac{4\mu_{\max}}{A}$, obtaining:

$$Y(t) = \frac{A}{1 + \exp(K_1\lambda - K_1 t + 2)}$$

$$\frac{dY(t)}{dt} = A \cdot \frac{K_1 \exp(K_1\lambda - K_1 t + 2)}{[1 + \exp(K_1\lambda - K_1 t + 2)]^2}$$

$$\frac{d^2Y(t)}{dt^2} = AK_1^2 \exp(K_1\lambda - K_1 t + 2)\frac{2\exp(K_1\lambda - K_1 t + 2) - [1 + \exp(K_1\lambda - K_1 t + 2)]}{[1 + \exp(K_1\lambda - K_1 t + 2)]^3}.$$

Thus, for equation (2) to be valid in this model, we have:

$$\begin{aligned}
\frac{d^2Y(t)}{dt^2} = 0 &\Leftrightarrow 2\exp(K_1\lambda - K_1 t + 2) - [1 + \exp(K_1\lambda - K_1 t + 2)] = 0\\
&\Leftrightarrow 1 + \exp(K_1\lambda - K_1 t + 2) = 2\exp(K_1\lambda - K_1 t + 2)\\
&\Leftrightarrow 1 = \exp(K_1\lambda - K_1 t + 2)\\
&\Leftrightarrow K_1\lambda - K_1 t + 2 = 0\\
&\Leftrightarrow t = \frac{2}{K_1} + \lambda.
\end{aligned}$$

We have obtained $t_{ifx} = \frac{A}{2\mu_{\max}} + \lambda$. Using information from Table 3, we conclude $t_{ifx} = \frac{923997}{90700}$. Then, $y_{ifx} = 1.175$ and $y_0 = 0.097$. We have obtained $B = \frac{923997}{90700}$ and $C = 1.078$ in equation (30). With the values already present in the literature and the values of B and C we deduced, our interactive fuzzy solution for LMZ at 20º C can be written as:

$$Y(t) = \frac{2.35}{\exp\left[\frac{174}{47}\mu_{\max}\left(\frac{923997}{90700} - \frac{1.078}{\mu_{\max}} - t\right)\right] + 1}. \tag{32}$$

Being a fuzzy function, for each $\alpha \in [0, 1]$, $[Y(t)]^\alpha$ is an interval function. We present some graphics later in this text. First, we need to obtain our interactive fuzzy solution for GOM at 20º C.

For the GOM model, we define $K_2 = \frac{\mu_{\max} e}{A}$, obtaining:

$$\begin{aligned}
Y(t) &= A\exp\{-\exp[K_2(\lambda - t) + 1]\} \\
\frac{dY(t)}{dt} &= AK_2 \exp[K_2(\lambda - t) + 1] \exp\{-\exp[K_2(\lambda - t) + 1]\} \\
\frac{d^2Y(t)}{dt^2} &= AK_2^2 \exp[K_2(\lambda - t) + 1] \exp\{-\exp[K_2(\lambda - t) + 1]\} \cdot \{\exp[K_2(\lambda - t) + 1] - 1\}
\end{aligned}$$

Then, for equation (2) to be valid in this model, we have:

$$\begin{aligned}
\frac{d^2Y(t)}{dt^2} = 0 &\Leftrightarrow 1 = \exp[K_2(\lambda - t) + 1] \\
&\Leftrightarrow 0 = K_2(\lambda - t) + 1 \\
&\Leftrightarrow t - \lambda = \frac{1}{K_2} \\
&\Leftrightarrow t = \frac{1}{K_2} + \lambda
\end{aligned}$$

With data from Table 3 we can obtain $t_{ifx} \approx 3.65$, $y_{ifx} = 0.2064$ and $y_0 = 0.015$. Our fuzzy solution for GOM with 20º C can be written as

$$Y(t) = 2.42\exp\left\{-\exp\left[\left(\mu_{\max}\frac{e}{2.42}\right) + (4.65 - \frac{0.191426}{\mu_{\max}} - t)\right]\right\}. \tag{33}$$

Graphics of fuzzy solutions

Now, we compare the graphics of our solutions. We present the graphics of $[Y(t)]^\alpha$ in each model with $\alpha = 0$, $\alpha = 0.5$, and $\alpha = 1$, representing respectively, the maximum diameter of the solution, the intermediary diameter of the solution and the minimum diameter of the solution, in the last case being the classical solution. Our method to extend the empirical solution to a fuzzy function is taking $\mu_{\max}$ as a fuzzy number, centered in the actual value as in Table 3, and with diameter such that we have the same 95% of confidence interval obtained empirically (Longhi et al., 2013).

The equation (32) when $\mu_{\max}$ is the triangular fuzzy number given by $(0.1502, 0.1814, 0.2125)$ is represented in Figure 1.

The equation (33) when $\mu_{\max}$ is the triangular fuzzy number given by $(0.1696, 0.1766, 0.1836)$ is represented in Figure 2.

While $\alpha = 1$, we have the deterministic solution, with the deterministic value also obtained empirically for each parameter. As α approaches 0, the α-levels of fuzzy solutions expand, as they begin to encompass larger and larger deviations from the parameters. When $\alpha = 0$, we obtain the maximum diameter predicted in the model, encompassing the maximum deviations that the confidence interval in the literature predicts for the parameter we make fuzzy.

Figure 1 - Graphics of $[Y(t)]^\alpha$ in LMZ for $\alpha = 0$, $\alpha = 0.5$, and $\alpha = 1$.

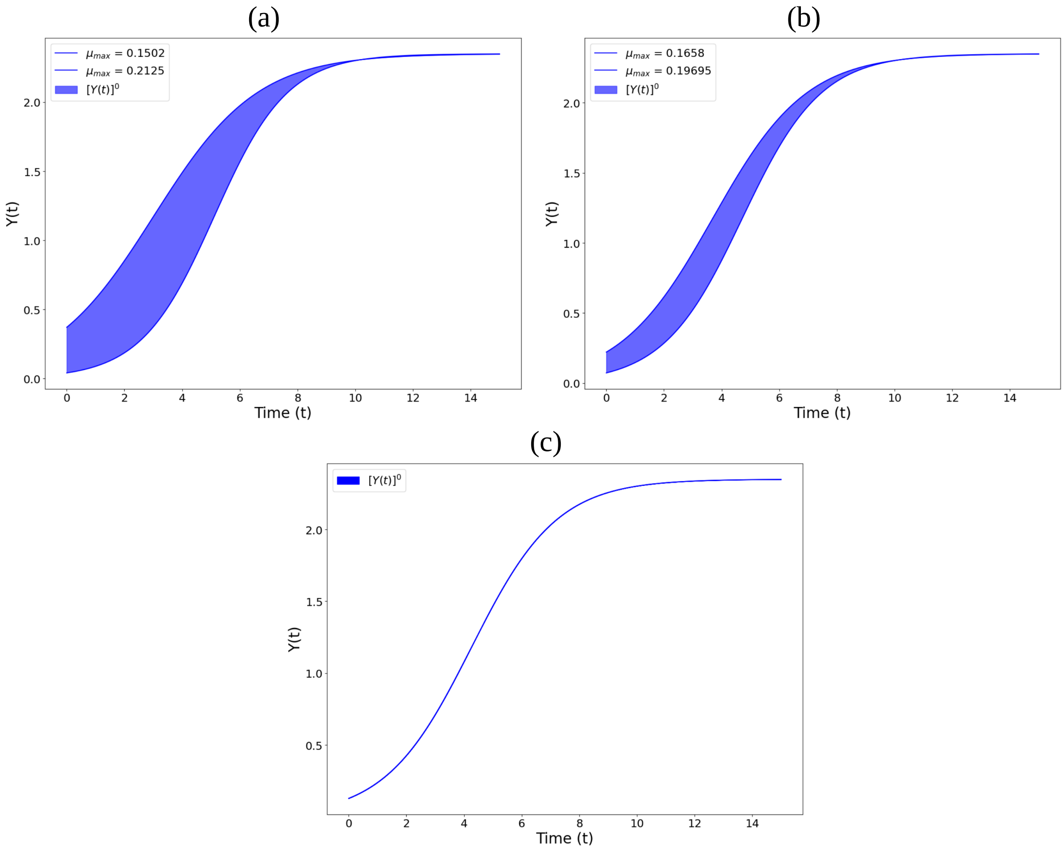
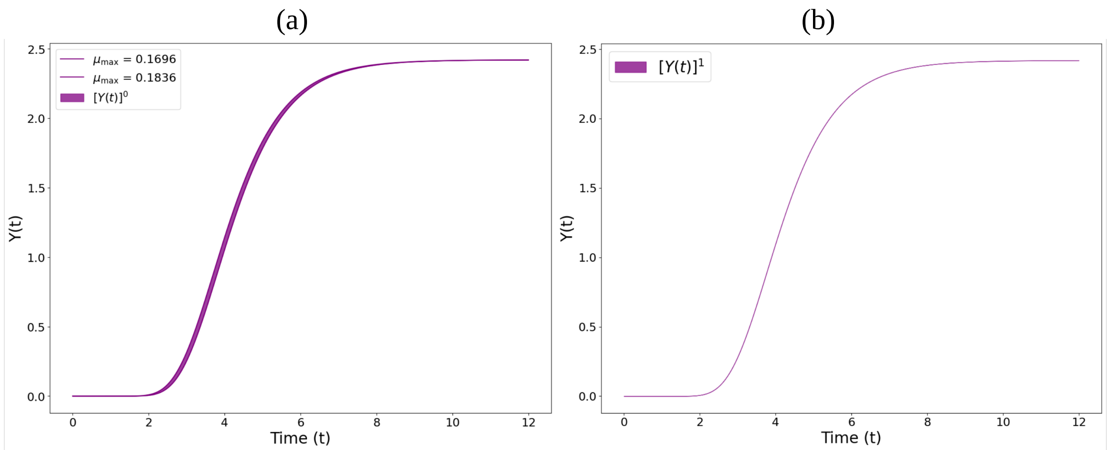


Figure 2 - Graphics of $[Y(t)]^\alpha$ in GOM for $\alpha = 0$ and $\alpha = 1$



Conclusions

We present fuzzy extensions to the solutions obtained in the literature for LMZ and GOM applied to the growth of microbial populations. Through the possible deviations of a parameter, we have deduced possible deviations of the entire model. Some empirical tests could verify the predictive power of our extended solutions, but it is certainly greater than the solutions represented by a single curve. The value of α for each level represents degrees of possible deviations; when α is close to 1, we have a more ideal (and therefore smaller) deviation.

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Author contributions

D. S. Silva participated in: Formal Analysis, Visualization, Software and Writing, preparation of the original draft. **R. A. C. Prata** participated in: Conceptualization, Supervision and Writing, revision & editing.

Conflicts of interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

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