Comparative analysis between measures of strain in structures of bar

Análise comparativa entre medidas de deformação em estruturas de barra

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Abstract

The performance of structures is of fundamental importance within the structural design, however, the structure response when subjected to loading is a function of several variables. Structures are designed to meet ruin safety criteria and meet service conditions so that unacceptable damage to nonstructural elements does not occur and to ensure the effective part durability. The contribution of computational systems assists the analysis process, especially in the study of complex structural systems. Thus, this paper aims to apply concepts of continuum mechanics through four one-dimensional strain measurements on a bar requested by an axial tensile load and to verify their behavior in the face of increased load. Cauchy's, Green's, Logarithmic and Almansi's strains were studied. It is concluded that, for small load increments, the difference between the numerical results obtained by the Ftool program and those obtained here were considerably close. However, with the consequent increase in loading, there was a large variation in the response of the bar regarding the displacement for the strain measurements. The displacements in the bar were obtained with the finite element positional method.

Keywords: Ftool. Strain measurements. Continuous mechanics. Finite element positional method.

Resumo

O desempenho de estruturas é de fundamental importância dentro do projeto estrutural, contudo a resposta da estrutura quando submetida a carregamentos é função de diversas variáveis. As estruturas são projetadas para atender a critérios de segurança contra ruína e satisfazer condições de serviço, para que não ocorra danos inaceitáveis em elementos não estruturais e garantam a efetiva durabilidade da peça. A contribuição de sistemas computacionais auxilia o processo de análise, principalmente no estudo de sistemas estruturais complexos. Desse modo, este artigo tem por objetivo aplicar conceitos da mecânica do contínuo por meio de quatro medidas de deformação unidimensional em uma barra carregada axialmente à tração e verificar o comportamento das mesmas frente ao aumento do carregamento. Foram estudadas as deformações de Cauchy, de Green, Logarítmica e de Almansi. Conclui-se que, para pequenos incrementos de carga, a diferença entre os resultados numéricos obtidos pelo programa Ftool e os obtidos aqui foram consideravelmente próximos. No entanto, com o consequente aumento do carregamento houve uma grande variação da resposta da barra quanto ao deslocamento para as medidas de deformação. Os deslocamentos na barra foram obtidos com o método posicional de elementos finitos.

Palavras-chave: Ftool. Medidas de deformação. Mecânica do contínuo. Método posicional de elementos finitos.

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Introduction

The use of deformable and elastic structural elements has been increasingly widespread, such as natural or vulcanized filled polymers. These materials contribute to reduced vibration, shock absorption and increased flexibility in connections.

One of the important characteristics of the materials is their ability to resist or transmit stresses. Their response under stress is closely related to the property of the material to deform elastically or plastically. A material exhibits elastic behavior when subjected to mechanical stress, it exhibits non-permanent deformations. When removing such stresses, the material returns to its original dimensions. The plastic behavior is observed when the same material is subjected to higher stresses and its dimensions are permanently changed, and once the efforts stop, the material does not return to its original dimensions (YAW, 2017; MELO, 2015).

The appearance of permanent deformations in the idealized medium corresponds to the process of disagreement movement in the crystalline structure of the materials. The mechanical response of materials to deformation occurs in part immediately and over time. The second has a higher or lower intensity according to the characteristics of the request and the medium itself.

In structural design, a correct assessment of the structure behavior is extremely important, and its response to a given load is a function of several random variables that affect its performance. When a system is subjected to extreme loading, it can undergo large displacements and deformations, and such behavior generally leads to partial or total collapse of the structure (WRIGGERS, 2008).

Structures are designed to meet ruin safety criteria and meet service conditions so that there is not unacceptable damage to non-structural elements under loading action. In addition, the degree of cracking must be within the acceptable range on the flexed parts so as not to affect the structure durability. According to Kimura (2007), when a serviceability limit state (SLS) and/or an ultimate limit state (ULS) is reached, the building use is unfeasible.

In the stability/instability analysis of structures, it is important, during the solution process, taking into account load and displacement limit points crossing in the equilibrium path (SANTANA, 2015). Computational methods are of great help in the process of modeling, analyzing and verifying the results, especially of structures with complex nonlinear behavior.

Several authors have sought analytical solutions of static equilibrium differential equations for mechanical problems involving small or large deformations. Timoshenko and Goodier (1951) described the static equilibrium equations, the geometric compatibility and the constitutive law for elastic and isotropic materials, subjected to small displacements and deformations. These authors presented analytical solutions for cases with simplified loading and geometry.

Schieck, Pietraszkiewicz and Stumpf (1992) showed the exact solution for highly deformable, elastic, circular plates with uniformly distributed transverse loading and embedded.

Coda and Greco (2004) developed the Finite Element positional formulation for geometric nonlinear static analysis of one-dimensional structures submitted to small or large deformations. This formulation is based on the principle of the minimum total potential energy and the fundamental unknowns of the problem are the positions of the finite element nodes, rather than the displacements, which are the unknowns in the standard finite element formulation for solids. The positional formulation is classified as a Total Lagrangian formulation (LACERDA, 2014).

Despite the great difference in the nature and internal structure of materials most commonly used in engineering, similar behavioral characteristics can be observed on a macroscopic scale (elasticity, viscosity, plastic deformation, etc.). Such behavioral condition justifies the use of Continuum Mechanics theory; although the hypothesis of medium continuum does not refer to the internal structure of the material, it plays an important role in the theoretical modeling, since it defines concepts such as stress and strain associated with material points (GONÇALVES, 2003).

Mathematical functions that represent the displacements and deformations suffered and that have the coordinates of the points as free variables could describe eventual transformations that lead to a configuration change. Continuum Mechanics provides several ways to measure one-dimensional strain in a structure, which are: Engineering's or Cauchy's strain; Green's strain; Logarithmic or Hencky's strain; and Almansi's strain.

In this context, this work aims to perform an analysis of the different strain measurements, from a problem consisting of an axially loaded bar to traction. The displacements in the bar were obtained with the positional finite element method.

Computational simulations are performed with Matlab software (MATLAB, 2015) and compared with the results obtained by the Ftool v.4.0 software (MARTHA, 1999; DEL SAVIO *et al.*, 2004). Analysis considered the constitutive relation of the linear elastic material and small displacements (hypotheses of linear analysis). The numerical results indicate the differences in the structural response with the strain measures.

Second section

One-dimensional strain measurements

The deformation of a bar is characterized by an offset u in the bar, as shown in Figure 1. L_0 is the initial length (before deformation) of the bar and L is its length after deformation, such that $u = L - L_0$ (MENIN, 2006; LACERDA, 2014).

Figure 1 – Bar elongation.
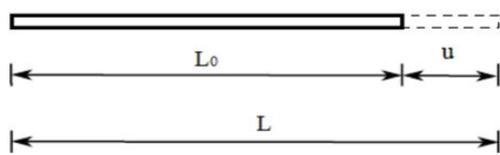
Source: Lacerda (2014).

Deformation is described in two senses: deformation and strain. In the sense of deformation as Ogden (1997), it is a body transformation from a reference configuration to the current configuration. Such a concept does not differ a simple rigid body movement from a change in body shape. Strain refers to a normalized dimensionless measure of the displacement between the material points of the body with respect to a reference configuration. Figure 2 illustrates such concepts.

Figure 2 – Difference between deformation and strain.
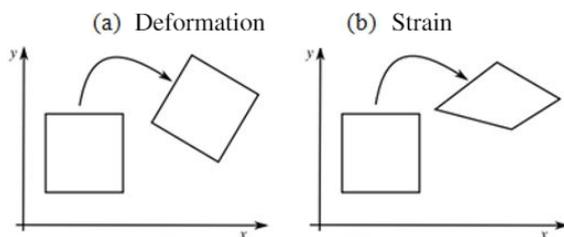
Source: Lacerda (2014).

Figure 2a illustrates a rigid body movement, and Figure 2b shows a movement in which changes body shape, not representing a rigid body. Both scenarios are considered deformation, i.e. configuration change. However, any strain measurement should

be null for the first situation and nonzero for the second (LACERDA, 2014). Several strain measurements may be appropriate for linear and nonlinear analysis.

Engineering's strain

Engineering's or Cauchy's strain (ε_e) is the simplest strain measurement, expressed by the following equation (MENIN, 2006; LACERDA, 2014):

$$\varepsilon_e = \frac{u}{L_0} = \frac{L - L_0}{L_0}, \qquad (1)$$

where L is the deformed length and L_0 is the initial or undeformed length.

Engineering's strain measures the deformation even though the bar has rotated greatly in the direction of ε_e.

Green's strain

Considering engineering's strain, it can be rewritten as follows (MENIN, 2006; LACERDA, 2014)

$$\varepsilon_e = \frac{(L - L_0)(L + L_0)}{L_0(L + L_0)}. \qquad (2)$$

As $L = u + L_0 = \varepsilon_e L_0 + L_0$, we have

$$\varepsilon_e = \frac{L^2 - L_0^2}{L_0(\varepsilon_e L_0 + L_0 + L_0)} = \frac{L^2 - L_0^2}{L_0^2(\varepsilon_e + 2)}. \qquad (3)$$

In case the strain is small, i.e. $\varepsilon_e \approx 0$, ε_e can be neglected in equation (3), obtaining the equation

$$\varepsilon_G = \frac{L^2 - L_0^2}{2L_0^2}, \qquad (4)$$

where ε_G is called Green's strain. There are many examples of structures with large displacements but with small strains. In these cases, Green's strain is suitable for the analysis of this type of structure.

Logarithmic strain

If the strain is too large, Logarithmic strain is more appropriate, also known as natural strain, true strain or Hencky's strain. The conceptualization of the theory consists of the sum of all infinitesimal strain increments that occur during the bar elongation, from the initial length L_0 to the final length L. The infinitesimal strain increment is given by (LACERDA, 2014)

$$d\varepsilon = \frac{dL}{L}. \qquad (5)$$

This increment integration is the definition of Logarithmic strain (ε_L), according to the equation

$$\varepsilon_L = \int_{L_0}^{L} d\varepsilon = \int_{L_0}^{L} \frac{dL}{L} = ln\left(\frac{L}{L_0}\right). \quad (6)$$

Even though Logarithmic strain can be generalized to more than one dimension, such generalization is complex and of high computational cost (BONET; WOOD, 2008).

Almansi's strain

The strain of Almansi (ε_A) is given by (MENIN, 2006; LACERDA, 2014)

$$\varepsilon_A = \frac{L^2 - L_0^2}{2L^2}. \quad (7)$$

The Almansi's strain design is similar to Green's strain. However, the difference is that the former refers to the deformed configuration and the latter to the initial configuration.

Positional formulation of finite elements for 2D truss bar

This section is a brief description of the positional formulation for 2D truss bar. The bar element only transmits axial forces and it has constant cross-sectional area A. Coordinates (X_1, Y_1) and (X_2, Y_2) represent the initial configuration of the bar element (also known as reference coordinates). Coordinates (x_1, y_1) and (x_2, y_2) represent the deformed configuration of the bar element.

Table 1 describes the internal force vector $\mathbf{F}_{el}$ and the stiffness matrix $\mathbf{K}_{el}$ for the element of 2D truss bar, obtained for the different strain measurements.

Table 1 – Describes the internal force vector $\mathbf{F}_{el}$ and the stiffness matrix $\mathbf{K}_{el}$ for the element of 2D truss bar, obtained for the different strain measurements.

Strain measurement	Internal force vector and stiffness matrix	
	$\mathbf{F}_{el}$	$\mathbf{K}_{el}$
Engineering	$\dfrac{EA\varepsilon_e}{L}\mathbf{b}$	$\dfrac{EA}{L^3}\mathbf{B}$
Green	$\dfrac{EA\varepsilon_G}{L_0}\mathbf{b}$	$\dfrac{EA}{L_0^3}\mathbf{B}$
Almansi	$\dfrac{EAL_0^2\varepsilon_A}{L^3}\mathbf{b}$	$\dfrac{EAL_0^2}{L^7}\left(L_0^2 - 3L^2\varepsilon_A\right)\mathbf{B}$
Logarithmic	$\dfrac{EAL_0\varepsilon_L}{L^2}\mathbf{b}$	$\dfrac{EAL_0}{L^4}\left(1 - 2\varepsilon_L\right)\mathbf{B}$

Source: Adapted from Lacerda (2014).

In Table 1, L_0 and L are the initial (reference) and deformed lengths of the bar element, respectively, E is the elastic modulus and $\mathbf{b}$ is the vector given by

$$\mathbf{b} = [x_1 - x_2 \quad y_1 - y_2 \quad x_2 - x_1 \quad y_2 - y_1]^T, \quad (8)$$

where x_i and y_i, with $i = 1, 2$, are the coordinates of node i of the bar, and matrix $\mathbf{B}$ is determined by

$$\mathbf{B} = \mathbf{bb}^T. \quad (9)$$

The displacement vector $\mathbf{u}$ is given by

$$\mathbf{u} = \mathbf{d} - {}^0\mathbf{d}, \quad (10)$$

where ${}^0\mathbf{d}$ is the nodal coordinate vector for the starting position of the bar (undeformed).

Methodology for solving structural problem

The basic structural problem is to find the equilibrium configuration of a structure that is under the action of applied forces. The equilibrium conditions of the finite elements representing the structure can be expressed by the following system of equations (BATHE, 2006)

$$\mathbf{g} = \mathbf{P} - \mathbf{F}_{int}(\mathbf{d}) = \mathbf{0}, \quad (11)$$

where $\mathbf{g}$ is unbalanced forces vector, $\mathbf{P}$ is the vector of nodal external forces and $\mathbf{F}_{int}$ is the global nodal internal forces vector corresponding to the finite element stresses - calculated as a function of the nodal coordinate vector $\mathbf{d}$. The solution of the equation system is solved by the Newton-Raphson method with constant load control. The iterative equations are

$$\mathbf{K}(\mathbf{d}^{(k-1)})(\delta\mathbf{d}^{(k)}) = \mathbf{P} - \mathbf{F}_{int}(\mathbf{d}^{(k-1)}), \quad (12)$$

$$\mathbf{d}^{(k)} = \mathbf{d}^{(k-1)} + \delta\mathbf{d}^{(k)}, \quad (13)$$

where $\mathbf{K}$ is called of global stiffness matrix and $\delta\mathbf{d}$ is the sub-increment of the nodal coordinate vector.

For a truss consisting of ne finite elements of bar, the matrix $\mathbf{K}$ and vector $\mathbf{F}_{int}$ are obtained from the stiffness matrix $\mathbf{K}_{el}$ and internal force vector $\mathbf{F}_{el}$ of each element, respectively, such that stated (BATHE, 2006)

$$\mathbf{K} = \sum_{i=1}^{ne} \mathbf{A}\mathbf{K}_{el_i}, \quad (14)$$

$$\mathbf{F}_{int} = \sum_{i=1}^{ne} \mathbf{A}\mathbf{F}_{el_i}, \quad (15)$$

where $\mathbf{A}$ is an assembly operator.

Numerical Results and Discussion

Being the model of an axially loaded bar as shown in Figure 3, which has one end with pinned-type support and the other with roller-type support, so that it has a single degree of freedom (x-axis displacement direction). The bar has initial length $L_0 = 1.0$ m, cross-sectional area $A = 1.0 \times 10^{-4}$ m^2 and elastic modulus $E = 10.0$ GPa. The structure is subjected to an axial load $P = 100.0$ kN assuming that it is applied slowly and in increments (static analysis).
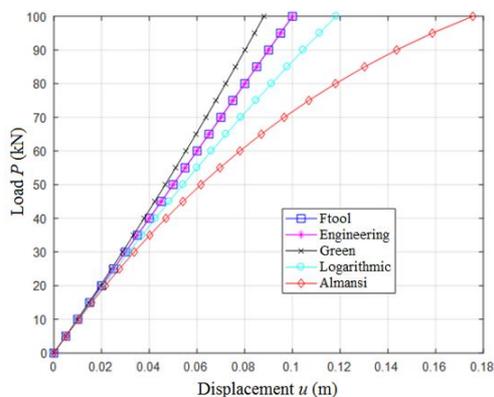
Figure 3 – Structural model of the bar.
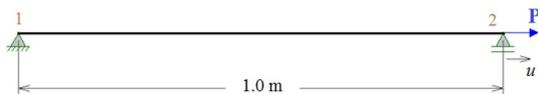
Source: The authors.

Figure 4 shows the equilibrium paths (load P *versus* horizontal displacement u). Table 2 presents the numerical results obtained for the load increments P, with the respective resulting strain measurements. The results are compared with those found by the Ftool v.4.0 software.

Figure 4 – Equilibrium paths.

Source: The authors.

The displacement found in Ftool served as a reference parameter with the strain measures implemented; Figure 4 shows that for lower intensity loads, which start the paths, the variation of the results is smaller. However, as the load increases, the strain measurements differ from the results found by the Ftool software, except for Engineering's strain, since this program makes use of this strain measurement, and the displacements coincide, as observed in Figure 4 and Table 2.

For loading up to 10.0 kN, the values found for all strain measurements are close to those obtained by Ftool, with a difference of less than 1.60 % for Green's strain and Logarithmic strain and less than 3.70 % Almansi's strain. Only Green's strain had a negative variation, with

values lower than those of Ftool, and the others occurred with positive variation and values higher than those found in this program.

When applied 100.0 kN loading, it is noticeable that the Almansi's strain provides the largest result difference of 75.57 %. Green's strain and Logarithmic strain portray the smallest variation in relation to Ftool , being 11.97 % and 18.33 %, respectively. Clearly, Green's strain and Logarithmic strain, although these measurements show a variation of less than 19.00 % when compared to Ftool, there is a considerable difference between them. Green's strain obtained the smallest difference between all measurements.

Depending on the load increment, Almansi's strain values generate a larger difference compared to the values observed in the reference parameter (Ftool); thus, larger loads provide greater displacements, and the strain measure tends to nonlinearity of the line, a situation that can be observed in the equilibrium path in Figure 4.

Conclusion

The results demonstrated the response of the bar requested to an axial tensile load as a function of the strain measures, whose material behavior is elastic-linear. The displacements obtained by Ftool were used as a reference parameter.

The strain measurements varied according to the load increase, and considering the characteristics of the numerical example, we concluded that for small load increments the measurements obtained results close to the Ftool, with little difference between the numerical values. However, the increase in loading generates a considerable difference depending on the amount of strain applied.

Logarithmic and Almansi's strain reached the largest variations when the maximum load was imposed, and for this study situation, such measures showed a relevant difference. In the case of Almansi's strain, we found the highest value inequality. Cauchy's or Engineering's strain resulted in exactly the same values to those obtained by Ftool.

Green's strain, compared to Logarithmic and Almansi's measurements, showed values closer to Ftool, even with increased loading.

The mechanical behavior of a given material is difficult to predict, requiring a more refined analysis, such as the consideration of physical and geometric nonlinearities, which brings results closer to experimental tests, in addition to the appropriate choice of strain measurement.

165

Table 2 – Numerical results for strain measurements

Load P (kN)	u (m)				
	Ftool	Engineering's strain	Green's strain	Logarithmic strain	Almansi's strain
5.0	0.005	0.005	0.004963	0.005038	0.005090
10.0	0.010	0.010	0.009854	0.010153	0.010367
15.0	0.015	0.015	0.014675	0.015347	0.015848
20.0	0.020	0.020	0.019430	0.020622	0.021548
25.0	0.025	0.025	0.024120	0.025981	0.027486
30.0	0.030	0.030	0.028748	0.031426	0.033684
35.0	0.035	0.035	0.033317	0.036960	0.040164
40.0	0.040	0.040	0.037827	0.042585	0.046957
45.0	0.045	0.045	0.042281	0.048304	0.054093
50.0	0.050	0.050	0.046681	0.054120	0.061610
55.0	0.055	0.055	0.051028	0.060035	0.069555
60.0	0.060	0.060	0.055324	0.066053	0.077980
65.0	0.065	0.065	0.059571	0.072177	0.086950
70.0	0.070	0.070	0.063770	0.078411	0.096545
75.0	0.075	0.075	0.067923	0.084758	0.106864
80.0	0.080	0.080	0.072031	0.091222	0.118034
85.0	0.085	0.085	0.076094	0.097806	0.130219
90.0	0.090	0.090	0.080115	0.104515	0.143642
95.0	0.095	0.095	0.084095	0.111353	0.158609
100.0	0.100	0.100	0.088034	0.118326	0.175571

Source: The authors.

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